Perturbation Analysis of Internal Balancing for Lightly Damped Mechanical Systems with Gyroscopic and Circulatory Forces

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Approximate expressions are developed for internally balanced singular values corresponding to the modes of mechanical systems with gryoscopic forces, light damping, and small circulatory forces. The singular values involve input and output coupling, modal frequency, and modal damping, and they serve as a guide for model reduction by modal truncation. The derivation of these singular values is based on perturbation analysis, and the satisfaction of a frequency separation condition is required to insure their validity.

Nomenclature

A	= system matrix
B	= input matrix
C	= output matrix
A_0	= system matrix for conservative forces
A_1	= system matrix for nonconservative forces
T	= transformation matrix
W_C	= system controllability grammian
W_0	= system observability grammian
$W_C^{(i,j)}$	= controllability grammian for two-mode subsystem
ϵ	= small parameter associated with nonconservative forces
σ_i	= ith balanced singular value
$\tilde{\sigma}_i$	= ith approximate balanced singular value
ω_i	= ith system frequency
ζ_i	= ith system damping ratio
Ω	= diagonal matrix of system frequencies

I. Introduction

As control theory grows, both the performance demanded and the size and complexity of the systems to be controlled are increasing rapidly. In the field of large space structures, very large, flexible systems must be controlled to extremely high accuracy. The mathematical models required to describe such systems are either infinite dimensional or have very large dimensions, usually too large to provide a tractable solution to a control problem. A number of model reduction schemes have been developed to deal with this problem, but one of the most appealing and widely used is the method of balanced realizations introduced by Moore.[1]

In general, obtaining a balanced realization involves the solution of two Lyapunov equations and an eigenvalue problem. The state variables that result from balancing may not have any physical interpretation. It is therefore advantageous to find a state-space representation that is easy to compute, has physical meaning, and is close to the balanced realization. Fortunately, it has been shown[2-5] that the modal representation for some systems becomes asymptotically balanced as damping approaches zero. This means that a lightly damped structure in modal form will be approximately balanced.

The present work also deals with the topic of balanced realizations for lightly damped systems. However, a different approach is used, and the class of systems considered is broader than that of Refs. 2–5. Gyroscopic and circulatory forces are included, and it is not assumed that the damping matrix is diagonal. Needing only the assumption that the system frequencies are not repeated, we show that even with the preceding generalizations, the modal representation becomes asymptotically balanced as the damping and circulatory forces reduce to zero. The method also results in approximate balanced singular values that weight the importance of the modes. These do not necessarily order the modes in terms of frequency alone, but they are a simple function of input and output coupling as well as damping and frequency. The relationship among these factors turns out to be intuitively appealing.

The organization of the paper is as follows. First we give a brief overview of the balanced realization model reduction method, including a discussion of more recent work. Next we define the models that are considered in this paper and present a perturbation analysis to show that the modal representation becomes asymptotically balanced as damping reduces to zero. We also calculate the approximate balanced singular values. Finally, we present a simple example of a flexible, dual-spin spacecraft that illustrates an application of these results.

II. Balanced Realization Model Reduction

Consider a state-space representation of a stable transfer function $G(s) = C(sI\text{-}A)^{-1}B$:

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

The associated controllability and observability grammians are

$$W_c = \int_0^\infty e^{At}BB^Te^{A^Tt}\,\mathrm{d}t, \qquad W_0 = \int_0^\infty e^{A^Tt}C^TCe^{At}\,\mathrm{d}t \tag{2}$$

W_c and W_o are solutions to the following Lyapunov equations:

$$AW_c + W_cA^T + BB^T = 0, \qquad A^TW_0 + W_0A + C^TC = 0 \tag{3}$$

There exists a linear transformation of variables, $x = Tz$, such that Eq. (1) can be written as

$$\dot{z} = T^{-1}ATz + T^{-1}Bu, \qquad y = CTz \tag{4}$$

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and

$$\bar{W}_c = T^{-1} W_c T^{-T} = \bar{W}_0 = T^T W_0 T = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \cdot \; \sigma_3 \end{bmatrix} \tag{5}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

The coordinates z are referred to as the "balanced coordinates" and the state-space realization defined by Eq. (4) as the "balanced realization." A reduced-order model of order k is found by neglecting the states $z_{k+1}, \ldots, z_n$.

The motivation for model reduction in the balanced realization is that the motion of states corresponding to small σ_i's results in small output energy and simultaneously requires large input energy. These states are therefore relatively less controllable and observable than states corresponding to large σ_i's.

While the basis for truncating states in a balanced realization is intuitively pleasing, the resulting reduced-order model is not necessarily optimal in any norm. In particular, Kabamba[6,7] indicates via a simple counterexample that it is not optimal in the L_2 norm. Hyland and Bernstein[8] show that while under certain conditions the balanced reduced-order model is close to an extremal of the L_2-optimal model reduction problem, this may be a local maximum rather than a minimum. On the other hand, balanced model reduction is close to optimal in the Hankel and L_∞ norms.[9–11] In particular, the balanced reduced-order model satisfies $\|G(j\omega) - G_r(j\omega)\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$, while the optimal Hankel-norm reduced-order model satisfies $\|G(j\omega) - G_r(j\omega)\|_\infty \leq (\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$. The choice of one norm vs the other will depend on the application. Minimizing the L_∞ norm minimizes the L_2 output error for bounded L_2 inputs, while minimizing the L_2 norm minimizes the L_2 output error for impulse inputs. The purpose of this paper is to motivate a sensible scheme for the modal truncation of lightly damped flexible structures. In this case, the choice of norms is probably not crucial.

A great deal of work has appeared in the literature, extending both the theory and the application of the balanced realization model reduction method. An extension of the method to discrete-time systems has been considered by Pernebo and Silverman[12] and by Fernando and Nicholson.[13] Jonckheere and Silverman[14] have suggested a closed-loop version of the balanced realizations, based on diagonalizing the solutions to the control and estimation Ricatti equations. Some limitations of this method are pointed out by Yousuff and Skelton.[15] Shokoohi, Silverman, and Dooren[16] have developed a balanced realization for linear time-variable systems, and Verriest and Kailath[17] have considered the very general class of analytic, time-varying, linear systems, including unstable systems. Enns[10,11] has introduced a frequency-weighted balanced realization that allows the designer to specify frequency ranges in which he would like his reduced model to most closely match the full-order model. Davis and Skelton[18] and Liu and Anderson[19] also suggest modifications on the standard balancing approach.

The result that the modal representation is approximately balanced is originally due to Jonckheere and Silverman.[3] Under some restrictive assumptions, they show that the balanced approximation and the optimal Hankel-norm approximation are equivalent to modal truncation in an asymptotic sense as damping is reduced to zero. They do not, however, consider the effect of actuator and sensor location or different degrees of damping in the modes, which can lead to a situation where a higher frequency mode will be considerably more important than a lower frequency mode. Jonckheere and Opdenaker[4,5] use a parameterization of balanced SISO systems to show the same results. They then treat displacement and rate output cases separately to derive approximate singular values. Finally, Gregory[2] deals with a more general system, though still not including gyroscopic or circulatory forces. In this case, he derives approximate singular values for simultaneous displacement and velocity measurements as well as an error bound based on frequency separation. Our results are a small generalization of Gregory's and reduce to his in the case where gyroscopic and circulatory forces are not present and damping does not couple modes.

III. Perturbation Analysis

Consider the general class of flexible systems represented by the following second-order model:

$$M\ddot{q} + G\dot{q} + Kq + \epsilon(D\dot{q} + Fq) = Bu, \qquad y = C_1 q + C_2 \dot{q} \tag{6}$$

where

- M = symmetric, positive definite mass matrix
- G = skew symmetric gyroscopic matrix
- K = symmetric semipositive definite stiffness matrix
- ϵ = small parameter indicative of light damping ($\epsilon > 0$)
- D = symmetric damping matrix
- F = skew symmetric circulatory matrix
- B = control input matrix
- u = input vector
- y = output vector
- C_1 = displacement measurement matrix
- C_2 = velocity measurement matrix

The gyroscopic matrix typically arises when the system contains rotating components. Circulatory forces of the form shown in Eq. (6) arise when the damping forces for rotating components are described in nonrotating reference frames. The two terms multiplied by ϵ are both nonconservative. The case of a flexible structure with neither gyroscopic nor circulatory forces is an important special case. Gregory[2] has calculated the approximate balanced singular values for this case, with the further assumption that damping does not couple the modes.

Define a modal representation of Eq. (6) to be one with the following state-space representation:

$$\dot{x} = (A_0 + \epsilon A_1)x + Bu, \qquad y = Cx \tag{7}$$

where

$$A_0 = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix}$$

$$B = [B_1 \quad B_2]^T, \qquad C = [C_1 \quad C_2]$$

$\Omega = \text{diag}\,\{\omega_i\}$ is the diagonal matrix of natural frequencies. An important special case occurs when there are no gyroscopic or circulatory terms (i.e., $G = F = 0$). Furthermore, if the damping matrix D is proportional to the stiffness matrix K (i.e., $D = K^\alpha$, $0 \leq \alpha \leq 1$) a modal representation can be found for which $\Delta_{12} = \Delta_{21} = 0$, $B_1 = 0$, and $\Delta_{11} = \Delta_{22} = \text{diag}\,\{\zeta_i \omega_i\}$. Two common damping models that result in D being proportional to K are the constant damping model where $\alpha = 1/2$ and the viscoelastic damping model where $\alpha = 1$. These cases are examined in Ref. 2. In the more general case, the only requirement is that the elements of ϵA_1 be considerably smaller than the diagonal elements of A_0 and that A_0 contain no repeated frequencies. Systems described by Eq. (6) may be placed in modal form by a number of methods. Meirovitch[20,21] finds a modal representation for the conservative portion of the system ($\epsilon = 0$) from a $2n \times 2n$ symmetric generalized eigenvalue problem. Janssens[22] shows that in some special cases only an $n \times n$ eigenvalue problem needs to be solved. Alternately, an eigenvalue decomposition of a state-space representation of Eq. (6) may be carried out, though this requires the solution of $2n \times 2n$ nonsymmetrical eigenvalue problem.

A perturbation approach will show that as damping approaches zero, the modal representation Eq. (7) asymptotically approaches a balanced representation. This method is intuitively simple, doesn't depend on any more advanced mathematical techniques than some simple matrix properties, and easily gives the approximate singular values.

The balanced representation is characterized by the property that both the controllability grammian W_c and the observability grammian W_o are diagonal. In the standard formulation, the coordinates are scaled such that W_c and W_o are also equal, but any representation with diagonal grammians is balanced within a scaling. We shall show that as $\epsilon \to 0$, both grammians are dominated by diagonal terms, and in the limit the modal coordinates become identical to the balanced coordinates. The only other assumption is that the system has no repeated frequencies. Gregory[2] has noted that the approximation depends on the separation of frequencies in the original model. Intuitively, this is because the controllability and observability grammians are continuous functions of the system parameters, so whenever the frequencies of two lightly damped modes are close together, one mode will affect the observability/controllability of the other. As an illustration of this, consider a system consisting of two identical masses and springs attached to a single support whose motion can be considered as the prescribed input to the system. This model is not completely controllable since the two mass/spring subsystems cannot be excited independently. Any model reduction scheme that results in a minimal (controllable) realization will reduce the system to a single mode. On the other hand, any scheme based on modal truncation will treat the two modes identically, therefore retaining both. This indicates that model reduction schemes based on modal truncation will necessarily be limited in the case of identical (or very close) frequencies. In analyzing the conditions under which the approximate singular values are close to the true balanced singular values, we will indicate exactly how this requirement on the separation of frequencies effects the approximation.

First, consider the controllability grammian W_c to be a function of the small parameter ϵ appearing in Eq. (7). $W_c(\epsilon)$ will be the solution to the following Lyapunov equation:

$$(A_0 + \epsilon A_1)W_c(\epsilon) + W_c(\epsilon)(A_0 + \epsilon A_1)^T + BB^T = 0 \tag{8}$$

Expand $W_c(\epsilon)$ into the power series:

$$W_c(\epsilon) = \epsilon^{-1}W_{-1} + W_0 + \epsilon W_1 + \epsilon^2 W_2 + \cdots \tag{9}$$

To see that this is the correct power series expansion, let $W_c(\epsilon) = \epsilon^\alpha W(\epsilon)$, where $W(\epsilon) = 0[1]$ as $\epsilon \to 0$. Then

$$\epsilon^\alpha[A_0 W(\epsilon) + W(\epsilon)A_0^T] + \epsilon^{\alpha+1}[A_1 W(\epsilon) + W(\epsilon)A_1^T] + BB^T = 0 \tag{10}$$

and α will satisfy one of the following: $\alpha < -1$, $\alpha = -1$, $-1 < \alpha < 0$, $\alpha = 0$, or $\alpha > 0$.

If a, c, or e are the case, we would have

$$A_0 W(\epsilon) + W(\epsilon)A_0^T = 0$$

$$A_1 W(\epsilon) + W(\epsilon)A_1^T = 0$$

$$BB^T = 0$$

where the last equation is a contradiction, so $\alpha = -1$ or $\alpha = 0$. We know that $W_c \uparrow \infty$ as $\epsilon \downarrow 0$, so $\alpha = -1$ is clearly the correct answer. This can be shown more rigorously by partitioning $W(\epsilon)$ and performing some simple algebra.

Now substitute Eq. (9) into Eq. (8) and separate in terms of powers of ϵ:

$$\epsilon^{-1}: A_0 W_{-1} + W_{-1}A_0^T = 0 \tag{11a}$$

$$\epsilon^0: A_0 W_0 + W_0 A_0^T + A_1 W_{-1} + W_{-1}A_1^T + BB^T = 0 \tag{11b}$$

$$\epsilon^1: A_0 W_1 + W_1 A_0^T + A_1 W_0 + W_0 A_1^T = 0 \tag{11c}$$

etc. Partition W_{-1} as

$$W_{-1} = \begin{bmatrix} w_{11} & w_{12} \\ w_{12}^T & w_{22} \end{bmatrix} \tag{12}$$

Substituting Eqs. (9) and (12) into Eq. (11a) results in the conclusion that $w_{12} = w_{21} = 0$ and $w_{11} = w_{22} = \mathrm{diag}\,\{w_{ci}\}$.

W_{-1} is therefore diagonal for the modal representation. Since this term will dominate as damping approaches zero, the controllability grammian will asymptotically become diagonal. Next, use Eq. (11b) to find the values of the diagonal entries of W_{-1}. Partition W_0 as

$$W_0 = \begin{bmatrix} u_{11} & u_{12} \\ u_{12}^T & u_{22} \end{bmatrix} \tag{13}$$

Substitute Eqs. (7) and (13) into Eq.(11b) and consider the diagonal terms to conclude that

$$w_{11ii} = w_{22ii} \triangleq w_{ci} = [(B_1B_1^T)_i + (B_2B_2^T)_i]/[2(\Delta_{11_i} + \Delta_{22_i})]; \quad i = 1, \ldots, n \tag{14}$$

Note that the entry corresponding to the position component of a mode (w_{11}) is identical to the entry corresponding to the velocity component of a mode (w_{22}). This indicates that the ordering of modes will always keep the position and velocity components of a given mode together.

An identical analysis can be done for the observability grammian, this time with the result that

$$w'_{11ii} = w'_{22ii} \triangleq w_{oi} = [(C_1^TC_1)_i + (C_2^TC_2)_i]/[2(\Delta_{11_i} + \Delta_{22_i})]; \quad i = 1, \ldots, n \tag{15}$$

The balanced singular values are the square roots of the products of the eigenvalues of the controllability and observability grammians, so the approximate singular value corresponding to both the position and velocity components of a mode is

$$\tilde{\sigma}_i = \frac{\sqrt{[(B_1B_1^T)_i + (B_2B_2^T)_i][(C_1^TC_1)_i + (C_2^TC_2)_i]}}{2\epsilon(\Delta_{11_i} + \Delta_{22_i})} \tag{16}$$

If the conditions $G = F = 0$ and $D = K^\alpha$ are satisfied, then this expression reduces to

$$\tilde{\sigma}_i = \frac{\sqrt{[(B_2B_2^T)_i][(C_1^TC_1)_i + (C_2^TC_2)_i]}}{4\zeta_i\omega_i} \tag{17}$$

which is equivalent to the expression for approximate singular values derived in Ref 2. The derivation in Ref. 2, however, assumes that the damping matrix is diagonal. The present development shows that this assumption is unnecessary.

Next, analyze the error in the approximate singular values of Eq. (16) by examining the next term in the expansion W_0 (see Eq. (9)). Using methods similar to those previously described, we find that

$$u_{12ij} = [(B_1B_1^T)_{ij}\omega_j + (B_2B_2^T)_{ij}\omega_i - (\Delta_{11ji}\omega_j + \Delta_{22ji}\omega_i)w_{ci} - (\Delta_{11ij}\omega_j + \Delta_{22ij}\omega_i)w_{cj}]/(\omega_i^2 - \omega_j^2)$$

$$u_{11ij} = [-(B_1B_2^T)_{ij}\omega_j + (B_2B_1^T)_{ij}\omega_i + (\Delta_{12ji}\omega_i - \Delta_{21ji}\omega_j)w_{ci} + (\Delta_{21ij}\omega_i - \Delta_{12ij}\omega_j)w_{cj}]/(\omega_i^2 - \omega_j^2)$$

$$u_{22ij} = [-(B_1B_2^T)_{ij}\omega_i + (B_2B_1^T)_{ij}\omega_j + (\Delta_{21ji}\omega_i - \Delta_{12ji}\omega_j)w_{ci} + (\Delta_{12ij}\omega_i - \Delta_{21ij}\omega_j)w_{cj}]/(\omega_i^2 - \omega_j^2) \tag{18}$$

Table 1 System model data

$$M = \begin{bmatrix} 2500.0 & 0 & 0 & 0 & 5.2903 & 0 & 0 & -0.63402 \\ & 8000.0 & -54.714 & 0 & 0 & 1.3132 & 0 & 0 \\ & & 0.49792 & 0 & 0 & 0 & 0 & 0 \\ & & & 0.18246 & 0 & 0 & 0 & 0 \\ & & & & 0.06500 & 0 & 0 & 0 \\ & \text{symmetric} & & & & 0.00981 & 0 & 0 \\ & & & & & & 0.00966 & 0 \\ & & & & & & & 0.00544 \end{bmatrix}$$

$$G = \left[\begin{array}{cc|c} 0 & 12566.4 & \\ & & 0_{2\times 6} \\ -12566.4 & 0 & \\ \hline \multicolumn{2}{c|}{0_{6\times 2}} & 0_{6\times 6} \end{array}\right] \quad K = \left[\begin{array}{c|c} 0_{2\times 2} & 0_{2\times 6} \\ \hline 0_{6\times 2} & 2_{6\times 6} \end{array}\right] \quad B = \left[\begin{array}{cc} 100.0 & 0 \\ 0 & 100.0 \\ \hline \multicolumn{2}{c}{0_{6\times 2}} \end{array}\right] \quad C = \left[\begin{array}{cc|c} 57.30 & 0 & \\ & & 0_{2\times 6} \\ 0 & 57.30 & \end{array}\right]$$

Table 2 Modal data

ζ_i (%)	ω_i (rad/s)	$\tilde{\sigma}_i$	σ_i
0.000×10^{0}	0.000×10^{0}	Infinite	Infinite
5.723×10^{-1}	1.677×10^{0}	1.175×10^{1}	$(1.181/1.169) \times 10^{1}$
1.873×10^{0}	7.725×10^{0}	5.779×10^{-1}	$(5.820/5.713) \times 10^{-1}$
2.261×10^{0}	5.266×10^{0}	1.451×10^{-1}	$(1.444/1.434) \times 10^{-1}$
7.406×10^{0}	1.511×10^{1}	2.859×10^{-3}	$(2.858/2.706) \times 10^{-3}$
9.774×10^{0}	1.949×10^{1}	1.017×10^{-3}	$(8.290/7.434) \times 10^{-4}$
1.656×10^{0}	3.310×10^{0}	2.496×10^{-31}	[a]
7.213×10^{0}	1.435×10^{1}	6.498×10^{-33}	[a]

[a]True balanced singular values are approximately zero.

To quantify the error, we examine a two-mode subsystem (i.e., a system of any two modes from Eq. (7). The controllability grammian for the subsystem is

$$W_C^{(i,j)} = \begin{bmatrix} \tilde{\sigma}_i & 0 & (u_{11})_{ij} & (u_{12})_{ij} \\ 0 & \tilde{\sigma}_i & (u_{12})_{ji} & (u_{22})_{ij} \\ (u_{11})_{ji} & (u_{12})_{ji} & \tilde{\sigma}_j & 0 \\ (u_{12})_{ij} & (u_{22})_{ji} & 0 & \tilde{\sigma}_j \end{bmatrix} \tag{19}$$

Denoting the maximum error in the eigenvalues of Eq. (19) by δ and applying a bound on this error,[23] we calculate

$$\frac{|\delta|}{\sigma_1\sigma_j} \le \frac{2\epsilon}{|\omega_i^2 - \omega_j^2|} \max(\omega_i, \omega_j) \times \max\{|\sqrt{(\Delta_{11}+\Delta_{22})_{ii}(\Delta_{11}+\Delta_{22})_{jj}} - (\Delta_{11}+\Delta_{22})_{ij}|, |(\Delta_{12}-\Delta_{21})_{ij}|\} \tag{20}$$

An analysis of the observability grammian will give identical results. From expressions like Eq. (19) for a two-mode subsystem, one can argue that the approximate balanced singular values are accurate if

$$\frac{\epsilon}{|\omega_i^2 - \omega_j^2|} \max(\omega_i, \omega_j) \max\{|\sqrt{(\Delta_{11}+\Delta_{22})_{ii}(\Delta_{11}+\Delta_{22})_{jj}} - (\Delta_{11}+\Delta_{22})_{ij}|, |(\Delta_{12}-\Delta_{21})_{ij}|\} \ll 1, \quad \forall i,j \tag{21}$$

Eq. (21) indicates the conditions under which the modal representation is approximately balanced. The results in this section reduce to those of Ref. 2 in the special case where $G = 0$, $F = 0$, and D is proportional to K.

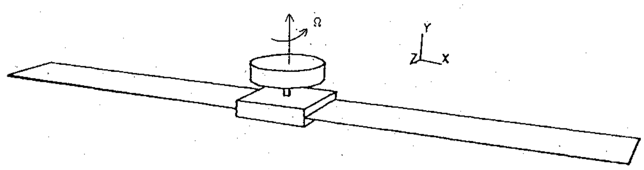

Fig. 1 Dual spin spacecraft.

IV. Example

Consider a dual-spin spacecraft consisting of a rigid platform with symmetric elastic panels and a rigid symmetric rotor spinning about an axis normal to the platform (Fig. 1). The modeling of this structure is described in Ref. 24, where the flexible motion of the panels is represented by the first two modes in in-plane bending, out-of-plane bending, and torsion, respectively. Since motion about the z axis decouples from the gyroscopic effects it is ignored, and the resulting model has eight modes: two rotational modes about the x and y axes respectively and the six flexible modes.

The results of Ref. 24 do not indicate any poles at the origin. The model should contain such poles since a rigid body displacement about either the x or y axis will not result in any restoring forces, and the spacecraft will remain in the displaced orientation. Therefore, while the motion of the spinning rotor does couple with angular velocities of the platform, it does not couple with angular displacements, and there will still be two poles at the origin. Since motion is described in body-fixed axes, the angular velocity of the rotor relative to the platform is a constant, and Eq. (53) of Ref. 24 should be modified to read

$$\omega_R = \Omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

This results in the following expression for kinetic energy, replacing Eq. (60) of Ref. 24 and using the notation of that reference:

$$T = \tfrac{1}{2}(A\dot{\theta}_1^2 + B\dot{\theta}_2^2) - \Omega C_R \theta_1 \dot{\theta}_2 - \dot{\theta}_2 \int (R + x)\dot{u}_z \, dm_E + \dot{\theta}_1 \int y\dot{u}_z \, dm_E - (\tfrac{1}{2}C)[\int (R + \dot{x})\dot{u}_y \, dm_E] + \tfrac{1}{2}\int (\dot{u}_y^2 + \dot{u}_z^2) \, dm_E \tag{23}$$

The potential energy is due only to structural stiffness, and this remains identical to that listed in Ref. 24. The matrices M, G, and K for Eq. (6) are listed in Table 1. These are identical to those listed in Ref. 24, except for changes in G and K due to the correction in the kinetic energy term. Damping is assumed to be viscoelastic, so the matrix D is ϵK where ϵ is chosen as 0.01. Assume a colocated actuator and sensor about the y axis, where the torque input is measured in N-m and the rotation is measured in degrees, resulting in the vectors B and C also listed in Table 1.

A modal analysis is accomplished by solving the following eigenvalue problem:

$$A(x_i + jy_i) = (\sigma_i + j\omega_i)(x_i + jy_i) \tag{24}$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}G - M^{-1}D \end{bmatrix}$$

The transformation matrix to the modal representation is

$$T = [x_1 \ldots x_n \,|\, y_1 \ldots y_n]$$

and the resulting state-space matrices are

$$T^{-1}AT = \begin{bmatrix} \Delta & \Omega \\ -\Omega & \Delta \end{bmatrix}, \qquad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = T^{-1}M^{-1}\begin{bmatrix} 0 \\ B \end{bmatrix}$$

and $[C_1 \quad C_2] = [C \quad 0]T$, where $\Delta = \mathrm{diag}\,\{\delta_i\}$.

The modal damping ratios, natural frequencies, approximate balanced singular values, and true balanced singular values are listed in Table 2. Though there is only one approximate balanced singular value corresponding to each mode, there are two true balanced singular values and both are listed. The approximate balanced singular values are close to the true ones, and they provide an ordering of the modes. As expected, the lower frequency, more lightly damped modes tend to be more important, though the procedure identifies two modes (one with a relatively low frequency) that are essentially uncontrollable and unobservable. These modes correspond to inplane bending motion that is not coupled to motion about the x or y axes, and their deletion will have no affect on the input/output model. In fact, if they are not removed from the system before the standard balancing algorithm is applied, significant numerical errors will appear. The true balanced singular values for these modes are therefore not calculated. The ordering also indicates that a mode near 8 rad/s is slightly more important than one near 5 rad/s, due to a higher degree of coupling to the inputs and outputs.

V. Conclusions

Approximate balanced singular values have been found to be an effective guide for the model reduction of lightly damped structural systems by modal truncation. The method has intuitive appeal, is computationally efficient, and permits the truncation to be performed in terms of a set of states that have physical significance. In the present paper, perturbation methods provide a systematic approach whereby previous results can be extended to systems with gyroscopic forces, circulatory forces, and damping that couples the modes.

The perturbation solution is valid when the nonconservative forces (damping and circulatory) are "small" and the system frequencies are "sufficiently separated." Expressions are developed, giving the frequency separation requirements. When the smallness and separation conditions are satisfied, the modal representation is approximately balanced in the sense of Moore.[1] For the special case where gyroscopic and circulatory forces are absent, the results reduce to those of Gregory.[2]

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